

WELL-POSEDNESS FOR THE CAUCHY PROBLEM OF THE KLEIN-GORDON-ZAKHAROV SYSTEM IN FIVE AND MORE DIMENSIONS

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ABSTRACT. We study the Cauchy problem of the Klein-Gordon-Zakharov system in spatial dimension $d \geq 5$ with initial datum $(u, \partial_t u, n, \partial_t n)|_{t=0} \in H^{s+1}(\mathbb{R}^d) \times H^s(\mathbb{R}^d) \times \dot{H}^s(\mathbb{R}^d) \times \dot{H}^{s-1}(\mathbb{R}^d)$. The critical value of s is $s_c = d/2 - 2$. By U^2, V^2 type spaces, we prove that the small data global well-posedness and scattering hold at $s = s_c$ in $d \geq 5$.

1. INTRODUCTION

We consider the Cauchy problem of the Klein-Gordon-Zakharov system:

$$\begin{cases} (\partial_t^2 - \Delta + 1)u = -nu, & (t, x) \in [-T, T] \times \mathbb{R}^d, \\ (\partial_t^2 - c^2 \Delta)n = \Delta|u|^2, & (t, x) \in [-T, T] \times \mathbb{R}^d, \\ (u, \partial_t u, n, \partial_t n)|_{t=0} = (u_0, u_1, n_0, n_1) \\ \qquad \qquad \qquad \in H^{s+1}(\mathbb{R}^d) \times H^s(\mathbb{R}^d) \times \dot{H}^s(\mathbb{R}^d) \times \dot{H}^{s-1}(\mathbb{R}^d), \end{cases} \quad (1.1)$$

where u, n are real valued functions, $d \geq 5, c > 0$ and $c \neq 1$. (1.1) describes the interaction of the Langmuir wave and the ion acoustic wave in a plasma. Physically, c satisfies $0 < c < 1$. When $d = 3$, Ozawa, Tsutaya and Tsutsumi [26] proved that (1.1) is globally well-posed in the energy space $H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3) \times L^2(\mathbb{R}^3) \times \dot{H}^{-1}(\mathbb{R}^3)$. They applied the Fourier restriction norm method to obtain the local well-posedness. Then by the local well-posedness and the energy method, they obtained the global well-posedness. For $d = 3$, Guo, Nakanishi and Wang [7] proved the scattering in the energy class with small, radial initial data. They applied the normal form reduction and the radial Strichartz estimates. If we transform $u_{\pm} := \omega_1 u \pm i\partial_t u, n_{\pm} := n \pm$

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$i(c\omega)^{-1}\partial_t n, \omega_1 := (1 - \Delta)^{1/2}, \omega := (-\Delta)^{1/2}$, then (1.1) is equivalent to the following.

$$\begin{cases} (i\partial_t \mp \omega_1)u_{\pm} = \pm(1/4)(n_+ + n_-)(\omega_1^{-1}u_+ + \omega_1^{-1}u_-), & (t, x) \in [-T, T] \times \mathbb{R}^d, \\ (i\partial_t \mp c\omega)n_{\pm} = \pm(4c)^{-1}\omega|\omega_1^{-1}u_+ + \omega_1^{-1}u_-|^2, & (t, x) \in [-T, T] \times \mathbb{R}^d, \\ (u_{\pm}, n_{\pm})|_{t=0} = (u_{\pm 0}, n_{\pm 0}) \in H^s(\mathbb{R}^d) \times \dot{H}^s(\mathbb{R}^d). \end{cases} \quad (1.2)$$

Our main result is as follows.

Theorem 1.1. *Let $d \geq 5, s = s_c = d/2 - 2$ and assume the initial data $(u_{\pm 0}, n_{\pm 0}) \in H^s(\mathbb{R}^d) \times \dot{H}^s(\mathbb{R}^d)$ is small. Then, (1.2) is globally well-posed in $H^s(\mathbb{R}^d) \times \dot{H}^s(\mathbb{R}^d)$.*

Corollary 1.2. *The solution obtained in Theorem 1.1 scatters as $t \rightarrow \pm\infty$.*

For more precise statement of Theorem 1.1 and Corollary 1.2, see Propositions 4.1, 4.2. [13] considered (1.2) for $d \geq 4, 0 < c$ and $c \neq 1$. [13] applied U^2, V^2 type spaces and obtained (1.2) is globally well-posed in $H^{s_c}(\mathbb{R}^d) \times \dot{H}^{s_c}(\mathbb{R}^d)$ if the initial data is small and radial. U^2, V^2 type spaces were introduced by Koch and Tataru [18]. These spaces works well as one consider well-posedness at the critical space [8], [11], [12], [14]. Theorem 1.1 is proved by the Banach fixed point theorem. The key is the bilinear estimate (Proposition 3.1). For $d \geq 5$, it seemed difficult to prove Proposition 3.1 only by applying U^2, V^2 type spaces, the modulation estimate (Proposition 2.12, Lemma 2.13) and the Strichartz type estimates (Proposition 2.8) for a nonlinear interaction [13]. In the present paper, to overcome the difficulty, we derive the bilinear Strichartz estimate for the nonlinear interaction and then we are able to prove Proposition 3.1. See Proposition 2.21 for the bilinear Strichartz estimate. $c \neq 1$ plays an important role in the proof of the bilinear Strichartz estimate as well as in the proof of Lemma 2.13.

In Section 2, we prepare some notations and lemmas with respect to U^p, V^p , in Section 3, we prove the bilinear estimates and in Section 4, we prove the main result.

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2. NOTATIONS AND PRELIMINARY LEMMAS

In this section, we prepare some lemmas, propositions and notations to prove the main theorem. $A \lesssim B$ means that there exists $C > 0$ such that $A \leq CB$. Also, $A \sim$

B means $A \lesssim B$ and $B \lesssim A$. Let $u = u(t, x)$. $\mathcal{F}_t u$, $\mathcal{F}_x u$ denote the Fourier transform of u in time, space, respectively. $\mathcal{F}_{t,x} u = \mathcal{F}u = \widehat{u}$ denotes the Fourier transform of u in space and time. Let \mathcal{Z} be the set of finite partitions $-\infty = t_0 < t_1 < \dots < t_K = \infty$ and let \mathcal{Z}_0 be the set of finite partitions $-\infty < t_0 < t_1 < \dots < t_K \leq \infty$.

Definition 1. Let $1 \leq p < \infty$. For $\{t_k\}_{k=0}^K \in \mathcal{Z}$ and $\{\phi_k\}_{k=0}^{K-1} \subset L_x^2$ with $\sum_{k=0}^{K-1} \|\phi_k\|_{L_x^2}^p = 1$, we call the function $a : \mathbb{R} \rightarrow L_x^2$ given by

$$a = \sum_{k=1}^K \mathbf{1}_{[t_{k-1}, t_k)} \phi_{k-1}$$

a U^p -atom. Furthermore, we define the atomic space

$$U^p := \left\{ u = \sum_{j=1}^{\infty} \lambda_j a_j \mid a_j : U^p\text{-atom}, \lambda_j \in \mathbb{C} \text{ such that } \sum_{j=1}^{\infty} |\lambda_j| < \infty \right\}$$

with norm

$$\|u\|_{U^p} := \inf \left\{ \sum_{j=1}^{\infty} |\lambda_j| \mid u = \sum_{j=1}^{\infty} \lambda_j a_j, \lambda_j \in \mathbb{C}, a_j : U^p\text{-atom} \right\}.$$

Proposition 2.1. *Let $1 \leq p < q < \infty$.*

- (i) U^p is a Banach space.
- (ii) The embeddings $U^p \subset U^q \subset L_t^\infty(\mathbb{R}; L_x^2)$ are continuous.
- (iii) For $u \in U^p$, it holds that $\lim_{t \rightarrow t_0+} \|u(t) - u(t_0)\|_{L_x^2} = 0$, i.e. every $u \in U^p$ is right-continuous.
- (iv) The closed subspace U_c^p of all continuous functions in U^p is a Banach space.

The above proposition is in [8] (Proposition 2.2).

Definition 2. Let $1 \leq p < \infty$. We define V^p as the normed space of all functions $v : \mathbb{R} \rightarrow L_x^2$ such that $\lim_{t \rightarrow \pm\infty} v(t)$ exist and for which the norm

$$\|v\|_{V^p} := \sup_{\{t_k\}_{k=0}^K \in \mathcal{Z}} \left(\sum_{k=1}^K \|v(t_k) - v(t_{k-1})\|_{L_x^2}^p \right)^{1/p}$$

is finite, where we use the convention that $v(-\infty) := \lim_{t \rightarrow -\infty} v(t)$ and $v(\infty) := 0$. Likewise, let V_-^p denote the closed subspace of all $v \in V^p$ with $\lim_{t \rightarrow -\infty} v(t) = 0$.

The definitions of V^p and V_-^p , see also [9].

Proposition 2.2. *Let $1 \leq p < q < \infty$.*

- (i) Let $v : \mathbb{R} \rightarrow L_x^2$ be such that

$$\|v\|_{V_0^p} := \sup_{\{t_k\}_{k=0}^K \in \mathcal{Z}_0} \left(\sum_{k=1}^K \|v(t_k) - v(t_{k-1})\|_{L_x^2}^p \right)^{1/p}$$

is finite. Then, it follows that $v(t_0^+) := \lim_{t \rightarrow t_0^+} v(t)$ exists for all $t_0 \in [-\infty, \infty)$ and $v(t_0^-) := \lim_{t \rightarrow t_0^-} v(t)$ exists for all $t_0 \in (-\infty, \infty]$ and moreover,

$$\|v\|_{V^p} = \|v\|_{V_0^p}.$$

- (ii) We define the closed subspace $V_{rc}^p(V_{-,rc}^p)$ of all right-continuous V^p functions (V_-^p functions). The spaces V^p , V_{rc}^p , V_-^p and $V_{-,rc}^p$ are Banach spaces.
- (iii) The embeddings $U^p \subset V_{-,rc}^p \subset U^q$ are continuous.
- (iv) The embeddings $V^p \subset V^q$ and $V_-^p \subset V_-^q$ are continuous.

The proof of Proposition 2.2 is in [8] (Proposition 2.4 and Corollary 2.6). Let $\{\mathcal{F}_\xi^{-1}[\varphi_n](x)\}_{n \in \mathbb{Z}} \subset \mathcal{S}(\mathbb{R}^d)$ be the Littlewood-Paley decomposition with respect to x , that is to say

$$\begin{cases} \varphi(\xi) \geq 0, \\ \text{supp } \varphi(\xi) = \{\xi \mid 2^{-1} \leq |\xi| \leq 2\}, \end{cases}$$

$$\varphi_n(\xi) := \varphi(2^{-n}\xi), \quad \sum_{n=-\infty}^{\infty} \varphi_n(\xi) = 1 \quad (\xi \neq 0), \quad \psi(\xi) := 1 - \sum_{n=0}^{\infty} \varphi_n(\xi).$$

Let $N = 2^n$ ($n \in \mathbb{Z}$) be dyadic number. P_N and $P_{<1}$ denote

$$\begin{aligned} \mathcal{F}_x[P_N f](\xi) &:= \varphi(\xi/N) \mathcal{F}_x[f](\xi) = \varphi_n(\xi) \mathcal{F}_x[f](\xi), \\ \mathcal{F}_x[P_{<1} f](\xi) &:= \psi(\xi) \mathcal{F}_x[f](\xi). \end{aligned}$$

Similarly, let \tilde{Q}_N be

$$\mathcal{F}_t[\tilde{Q}_N g](\tau) := \phi(\tau/N) \mathcal{F}_t[g](\tau) = \phi_n(\tau) \mathcal{F}_t[g](\tau),$$

where $\{\mathcal{F}_\tau^{-1}[\phi_n](t)\}_{n \in \mathbb{Z}} \subset \mathcal{S}(\mathbb{R})$ be the Littlewood-Paley decomposition with respect to t . Let $K_\pm(t) = \exp\{\mp it(1 - \Delta)^{1/2}\} : L_x^2 \rightarrow L_x^2$ be the Klein-Gordon unitary operator such that $\mathcal{F}_x[K_\pm(t)u_0](\xi) = \exp\{\mp it\langle \xi \rangle\} \mathcal{F}_x[u_0](\xi)$. Similarly, we define the wave unitary operator $W_{\pm c}(t) = \exp\{\mp ict(-\Delta)^{1/2}\} : L_x^2 \rightarrow L_x^2$ such that $\mathcal{F}_x[W_{\pm c}(t)n_0](\xi) = \exp\{\mp ict|\xi|\} \mathcal{F}_x[n_0](\xi)$. We set

$$\begin{aligned} W_L^{\pm c} &:= \{(\tau, \xi) \in \mathbb{R} \times \mathbb{R}^d \mid L/2 \leq |\tau \pm c|\xi| \leq 2L\}, \\ KG_L^\pm &:= \{(\tau, \xi) \in \mathbb{R} \times \mathbb{R}^d \mid L/2 \leq |\tau \pm \langle \xi \rangle| \leq 2L\}. \end{aligned}$$

Definition 3. We define

$$(i) \ U_{K_\pm}^p = K_\pm(\cdot)U^p \text{ with norm } \|u\|_{U_{K_\pm}^p} = \|K_\pm(-\cdot)u\|_{U^p},$$

(ii) $V_{K_\pm}^p = K_\pm(\cdot)V^p$ with norm $\|u\|_{V_{K_\pm}^p} = \|K_\pm(\cdot)u\|_{V^p}$.

For dyadic numbers N, M ,

$$Q_N^{K_\pm} := K_\pm(\cdot)\tilde{Q}_N K_\pm(\cdot), \quad Q_{\geq M}^{K_\pm} := \sum_{N \geq M} Q_N, \quad Q_{< M}^{K_\pm} := Id - Q_{\geq M}^{K_\pm}.$$

Here summation over N means summation over $n \in \mathbb{Z}$. Similarly, we define $U_{W_{\pm c}}^p, V_{W_{\pm c}}^p$.

Remark 2.1. For L_x^2 unitary operator $A = K_\pm$ or $W_{\pm c}$,

$$U_A^2 \subset V_{-,rc,A}^2 \subset L^\infty(\mathbb{R}; L_x^2)$$

Definition 4. For the Klein-Gordon equation, we define $Y_{K_\pm}^s$ (resp. $Z_{K_\pm}^s$) as the closure of all $u \in C(\mathbb{R}; H_x^s(\mathbb{R}^d)) \cap \langle \nabla_x \rangle^{-s} V_{-,rc,K_\pm}^2$ (resp. $u \in C(\mathbb{R}; H_x^s(\mathbb{R}^d)) \cap \langle \nabla_x \rangle^{-s} U_{K_\pm}^2$) with $Y_{K_\pm}^s$ (resp. $Z_{K_\pm}^s$) norm, where

$$\begin{aligned} \|u\|_{Y_{K_\pm}^s} &:= \|P_{<1}u\|_{V_{K_\pm}^2} + \left(\sum_{N \geq 1} N^{2s} \|P_N u\|_{V_{K_\pm}^2}^2 \right)^{1/2}, \\ \|u\|_{Z_{K_\pm}^s} &:= \|P_{<1}u\|_{U_{K_\pm}^2} + \left(\sum_{N \geq 1} N^{2s} \|P_N u\|_{U_{K_\pm}^2}^2 \right)^{1/2}. \end{aligned}$$

For the wave equation, we define $\dot{Y}_{W_{\pm c}}^s, \dot{Z}_{W_{\pm c}}^s$ as the closure of all $n \in C(\mathbb{R}; H_x^s(\mathbb{R}^d)) \cap |\nabla_x|^{-s} V_{-,rc,W_{\pm c}}^2$ (resp. $n \in C(\mathbb{R}; H_x^s(\mathbb{R}^d)) \cap |\nabla_x|^{-s} U_{W_{\pm c}}^2$) with $\dot{Y}_{W_{\pm c}}^s$ (resp. $\dot{Z}_{W_{\pm c}}^s$) norm, where

$$\|n\|_{\dot{Y}_{W_{\pm c}}^s} := \left(\sum_N N^{2s} \|P_N n\|_{V_{W_{\pm c}}^2}^2 \right)^{1/2}, \quad \|n\|_{\dot{Z}_{W_{\pm c}}^s} := \left(\sum_N N^{2s} \|P_N n\|_{U_{W_{\pm c}}^2}^2 \right)^{1/2}.$$

Definition 5. For a Hilbert space H and a Banach space $X \subset C(\mathbb{R}; H)$, we define

$$B_r(H) := \{f \in H \mid \|f\|_H \leq r\},$$

$$X([0, T]) := \{u \in C([0, T]; H) \mid \exists \tilde{u} \in X, \tilde{u}(t) = u(t), t \in [0, T]\}$$

endowed with the norm $\|u\|_{X([0, T])} = \inf\{\|\tilde{u}\|_X \mid \tilde{u}(t) = u(t), t \in [0, T]\}$.

We denote the Duhamel term

$$\begin{aligned} I_{T, K_\pm}(n, v) &:= \pm \int_0^t \mathbf{1}_{[0, T]}(t') K_\pm(t - t') n(t') (\omega_1^{-1} v(t')) dt', \\ I_{T, W_{\pm c}}(u, v) &:= \pm \int_0^t \mathbf{1}_{[0, T]}(t') W_{\pm c}(t - t') u((\omega_1^{-1} u(t')) (\overline{(\omega_1^{-1} v(t'))})) dt' \end{aligned}$$

for the Klein-Gordon equation and the wave equation respectively. The following proposition is in [8] (Theorem 2.8 and Proposition 2.10).

Proposition 2.3. *Let $u \in V_{-,rc}^1 \subset U^2$ be absolutely continuous on compact intervals.*

Then, $\|u\|_{U^2} = \sup_{v \in V^2, \|v\|_{V^2}=1} \left| \int_{-\infty}^{\infty} \langle u'(t), v(t) \rangle_{L_x^2} dt \right|$.

Corollary 2.4. *Let $A = K_{\pm}$ or $W_{\pm c}$ and $u \in V_{-,rc,A}^1 \subset U_A^2$ be absolutely continuous on compact intervals. Then,*

$$\|u\|_{U_A^2} = \sup_{v \in V_A^2, \|v\|_{V_A^2}=1} \left| \int_{-\infty}^{\infty} \langle A(t)(A(-\cdot)u)'(t), v(t) \rangle_{L_x^2} dt \right|.$$

Proposition 2.5. *Let $T_0 : L_x^2 \times \dots \times L_x^2 \rightarrow L_{loc}^1(\mathbb{R}^d; \mathbb{C})$ be a n -linear operator. Assume that for some $1 \leq p < \infty$ and $1 \leq q \leq \infty$, it holds that*

$$\|T_0(K_{\pm}(\cdot)\phi_1, \dots, K_{\pm}(\cdot)\phi_n)\|_{L_t^p(\mathbb{R}; L_x^q(\mathbb{R}^d))} \lesssim \prod_{i=1}^n \|\phi_i\|_{L_x^2}.$$

Then, there exists $T : U_{K_{\pm}}^p \times \dots \times U_{K_{\pm}}^p \rightarrow L_t^p(\mathbb{R}; L_x^q(\mathbb{R}^d))$ satisfying

$$\|T(u_1, \dots, u_n)\|_{L_t^p(\mathbb{R}; L_x^q(\mathbb{R}^d))} \lesssim \prod_{i=1}^n \|u_i\|_{U_{K_{\pm}}^p},$$

such that $T(u_1, \dots, u_n)(t)(x) = T_0(u_1(t), \dots, u_n(t))(x)$ a.e.

See Proposition 2.19 in [8] for the proof of the above proposition.

Proposition 2.6. *Let $d \geq 3, 2 \leq r < \infty, 2/q = (d-1)(1/2 - 1/r), (q, r) \neq (2, 2(d-1)/(d-3))$ and $s = 1/q - 1/r + 1/2$. Then it holds that*

$$\|W_{\pm c}(t)f\|_{L_t^q \dot{W}_x^{-s,r}(\mathbb{R}^{1+d})} \lesssim \|f\|_{L_x^2(\mathbb{R}^d)}.$$

For the proof of Proposition 2.6, see [15], [5].

Proposition 2.7. *Let $d \geq 3, 2 \leq r < \infty, 2/q = (d-1)(1/2 - 1/r), (q, r) \neq (2, 2(d-1)/(d-3))$ and $s = 1/q - 1/r + 1/2$. Then, it holds that*

$$\|K_{\pm}(t)f\|_{L_t^q W_x^{-s,r}(\mathbb{R}^{1+d})} \lesssim \|f\|_{L_x^2(\mathbb{R}^d)}.$$

For the proof of Proposition 2.7, see [22]. Combining Proposition 2.2, Proposition 2.5, Proposition 2.6 and Proposition 2.7, we have the following.

Proposition 2.8. *Let $d \geq 3, 2 \leq r < \infty, 2/q = (d-1)(1/2 - 1/r), (q, r) \neq (2, 2(d-1)/(d-3))$ and $s = 1/q - 1/r + 1/2$. If $p < q$, then it holds that*

$$\|f\|_{L_t^q W_x^{-s,r}(\mathbb{R}^{1+d})} \lesssim \|f\|_{V_{K_{\pm}}^p}, \quad \|f\|_{L_t^q \dot{W}_x^{-s,r}(\mathbb{R}^{1+d})} \lesssim \|f\|_{V_{W_{\pm c}}^p}.$$

Proposition 2.9. (i) *Let $T > 0$ and $u \in Y_{K_{\pm}}^s([0, T]), u(0) = 0$. Then, there exists $0 \leq T' \leq T$ such that $\|u\|_{Y_{K_{\pm}}^s([0, T'])} < \varepsilon$.*

(ii) *Let $T > 0$ and $n \in \dot{Y}_{W_{\pm c}}^s([0, T]), n(0) = 0$. Then, there exists $0 \leq T' \leq T$ such that $\|n\|_{\dot{Y}_{W_{\pm c}}^s([0, T'])} < \varepsilon$.*

For the proofs of (i) and (ii), see Proposition 2.24 in [8].

Lemma 2.10. *Let $a \geq 0$. Then for $A = K_{\pm}$ or $W_{\pm c}$, it holds that*

$$\|\langle \nabla_x \rangle^a f\|_{V_A^2} \lesssim \|f\|_{Y_A^a}.$$

Proof. We only prove for $A = K_{\pm}$ since we can prove similarly for $A = W_{\pm c}$. By L_x^2 orthogonality, we have

$$\begin{aligned} \|\langle \nabla_x \rangle^a f\|_{V_{K_{\pm}}^2}^2 &\lesssim \sup_{\{t_i\}_{i=0}^I \in \mathcal{Z}} \sum_{i=1}^I (\|P_{<1}(K_{\pm}(-t_i)f(t_i) - K_{\pm}(-t_{i-1})f(t_{i-1}))\|_{L_x^2}^2 \\ &\quad + \sum_{N \geq 1} N^{2a} \|P_N(K_{\pm}(-t_i)f(t_i) - K_{\pm}(-t_{i-1})f(t_{i-1}))\|_{L_x^2}^2) \\ &\lesssim \sup_{\{t_i\}_{i=0}^I \in \mathcal{Z}} \sum_{i=1}^I \|K_{\pm}(-t_i)P_{<1}f(t_i) - K_{\pm}(-t_{i-1})P_{<1}f(t_{i-1})\|_{L_x^2}^2 \\ &\quad + \sum_{N \geq 1} N^{2a} \sup_{\{t_i\}_{i=0}^I \in \mathcal{Z}} \sum_{i=1}^I \|K_{\pm}(-t_i)P_N f(t_i) - K_{\pm}(-t_{i-1})P_N f(t_{i-1})\|_{L_x^2}^2 \\ &\lesssim \|f\|_{Y_{K_{\pm}}^a}^2. \end{aligned}$$

□

Remark 2.2. Similarly, we see

$$\| |\nabla_x|^a f \|_{V_A^2} \lesssim \|f\|_{\dot{Y}_A^a}.$$

Lemma 2.11. *If f, g are measurable functions, then for $Q = Q_{<M}^A$ or $Q_{\geq M}^A$, $A = K_{\pm}$ or $W_{\pm c}$, it holds that*

$$\int_{\mathbb{R}^{1+d}} f(t, x) \overline{Qg(t, x)} dx dt = \int_{\mathbb{R}^{1+d}} (Qf(t, x)) \overline{g(t, x)} dx dt.$$

For the proof of Lemma 2.11, see [14], Lemma 2.17. Since $Q_{<M}^A = Id - Q_{\geq M}^A$, we also obtain the result for $Q = Q_{<M}^A$.

Proposition 2.12. *It holds that*

$$\begin{aligned} \|Q_M^{K_{\pm}} u\|_{L_{t,x}^2(\mathbb{R}^{1+d})} &\lesssim M^{-1/2} \|u\|_{V_{K_{\pm}}^2}, \quad \|Q_{\geq M}^{K_{\pm}} u\|_{L_{t,x}^2(\mathbb{R}^{1+d})} \lesssim M^{-1/2} \|u\|_{V_{K_{\pm}}^2}, \quad (2.1) \\ \|Q_{<M}^{K_{\pm}} u\|_{V_{K_{\pm}}^2} &\lesssim \|u\|_{V_{K_{\pm}}^2}, \quad \|Q_{\geq M}^{K_{\pm}} u\|_{V_{K_{\pm}}^2} \lesssim \|u\|_{V_{K_{\pm}}^2}, \\ \|Q_{<M}^{K_{\pm}} u\|_{U_{K_{\pm}}^2} &\lesssim \|u\|_{U_{K_{\pm}}^2}, \quad \|Q_{\geq M}^{K_{\pm}} u\|_{U_{K_{\pm}}^2} \lesssim \|u\|_{U_{K_{\pm}}^2}. \end{aligned}$$

The same estimates hold by replacing the Klein-Gordon operator K_{\pm} by the wave operator $W_{\pm c}$.

Lemma 2.13. *Let $c > 0, c \neq 1$ and $\tau_3 = \tau_1 - \tau_2$, $\xi_3 = \xi_1 - \xi_2$. If $|\xi_1| \gg \langle \xi_2 \rangle$ or $\langle \xi_1 \rangle \ll |\xi_2|$, then it holds that*

$$\max \{ |\tau_1 \pm \langle \xi_1 \rangle|, |\tau_2 \pm \langle \xi_2 \rangle|, |\tau_3 \pm c|\xi_3| | \} \gtrsim \max \{ |\xi_1|, |\xi_2| \}. \quad (2.2)$$

Proof. We only prove the case $|\xi_1| \gg \langle \xi_2 \rangle$ since the case $\langle \xi_1 \rangle \ll |\xi_2|$ is proved by the same manner.

$$(\text{l.h.s.}) \gtrsim |(\tau_1 \pm (1 + |\xi_1|)) - (\tau_2 \pm (1 + |\xi_2|)) - (\tau_3 \pm c|\xi_3|)| \quad (2.3)$$

If $0 < c < 1$, then we take ε_c such that $0 < \varepsilon_c < (1 - c)/(1 + c)$, $|\xi_2| \leq \varepsilon_c |\xi_1|$. Then, the right hand side of (2.3) is bounded by

$$(1 + |\xi_1|) - (1 + |\xi_2|) - c|\xi_1 - \xi_2| \geq |\xi_1| - \varepsilon_c |\xi_1| - c(1 + \varepsilon_c)|\xi_1| \gtrsim |\xi_1|.$$

If $c > 1$, then we take $\tilde{\varepsilon}_c$ such that $0 < \tilde{\varepsilon}_c < (c - 1)/(c + 3)$, $|\xi_2| \leq \tilde{\varepsilon}_c |\xi_1|$, $|\xi_1| \geq 1/\tilde{\varepsilon}_c$. Then, the right hand side of (2.3) is bounded by

$$c|\xi_1 - \xi_2| - (1 + |\xi_1|) - (1 + |\xi_2|) \geq c(1 - \tilde{\varepsilon}_c)|\xi_1| - (1 + \tilde{\varepsilon}_c)|\xi_1| - 2\tilde{\varepsilon}_c|\xi_1| \gtrsim |\xi_1|.$$

□

Remark 2.3. From (2.1) and (2.2), we can obtain a half derivative.

Lemma 2.14. *Let $\tilde{u}_{N_1} := \mathbf{1}_{[0,T)} P_{N_1} u$, $\tilde{v}_{N_2} := \mathbf{1}_{[0,T)} P_{N_2} v$, $\tilde{n}_{N_3} := \mathbf{1}_{[0,T)} P_{N_3} n$, $Q_1, Q_2 \in \{Q_{<M}^{K_{\pm}}, Q_{\geq M}^{K_{\pm}}\}$, $Q_3 \in \{Q_{<M}^{W_{\pm c}}, Q_{\geq M}^{W_{\pm c}}\}$. Let $s = s_c = d/2 - 2$. Then the following estimates hold for all $0 < T < \infty$:*

(i) *If $N_3 \lesssim N_2 \sim N_1$, then*

$$|I_1| := \left| \int_{\mathbb{R}^{1+d}} (\omega_1^{-1} \tilde{u}_{N_1}) (\overline{\omega_1^{-1} \tilde{v}_{N_2}}) (\overline{\omega \tilde{n}_{N_3}}) dx dt \right| \lesssim N_3^s \|u_{N_1}\|_{V_{K_{\pm}}^2} \|v_{N_2}\|_{V_{K_{\pm}}^2} \|n_{N_3}\|_{V_{W_{\pm c}}^2}.$$

(ii) *It holds that*

$$|I_2| := \left| \int_{\mathbb{R}^{1+d}} \tilde{n}(\omega_1^{-1} \tilde{v}) (\overline{P_{<1} \tilde{u}}) dx dt \right| \lesssim \|n\|_{\dot{Y}_{W_{\pm c}}^s} \|v\|_{Y_{K_{\pm}}^s} \|P_{<1} u\|_{V_{K_{\pm}}^2}.$$

(iii) *If $N_1 \sim N_2$, then*

$$|I_3| := \left| \int_{\mathbb{R}^{1+d}} \left(\sum_{N_3 \lesssim N_2} \tilde{n}_{N_3} \right) (\omega_1^{-1} \tilde{v}_{N_2}) \overline{\tilde{u}_{N_1}} dx dt \right| \lesssim \|n\|_{\dot{Y}_{W_{\pm c}}^s} \|v_{N_2}\|_{V_{K_{\pm}}^2} \|u_{N_1}\|_{V_{K_{\pm}}^2}.$$

(iv) *If $N_1 \sim N_3$, $N_1 \gg 1$, $M = \varepsilon N_1$ and $\varepsilon > 0$ is sufficiently small, then*

$$|I_i| \lesssim \|n_{N_3}\|_{V_{W_{\pm c}}^2} \|v\|_{Y_{K_{\pm}}^s} \|u_{N_1}\|_{V_{K_{\pm}}^2}, \quad (i = 4, 5)$$

where

$$I_4 := \int_{\mathbb{R}^{1+d}} (Q_{\geq M}^{W_{\pm c}} \tilde{n}_{N_3}) \left(\sum_{N_2 \ll N_1} Q_2 \omega_1^{-1} \tilde{v}_{N_2} \right) (\overline{Q_1 \tilde{u}_{N_1}}) dx dt,$$

$$I_5 := \int_{\mathbb{R}^{1+d}} (Q_3 \tilde{n}_{N_3}) \left(\sum_{N_2 \ll N_1} Q_2 \omega_1^{-1} \tilde{v}_{N_2} \right) (\overline{Q_{\geq M}^{K_{\pm}} \tilde{u}_{N_1}}) dx dt.$$

Proof. We show (i) first. For $f \in V_A^2$, $A \in \{K_{\pm}, W_{\pm c}\}$, we see

$$\|\mathbf{1}_{[0,T)} f\|_{V_A^2} \lesssim \|f\|_{V_A^2}. \quad (2.4)$$

For $d \geq 5$, we apply the Hölder inequality to have

$$|I_1| \lesssim \|\omega_1^{-1} \tilde{u}_{N_1}\|_{L_{t,x}^{2(d+1)/(d-1)}} \|\omega_1^{-1} \tilde{v}_{N_2}\|_{L_{t,x}^{2(d+1)/(d-1)}} \|\omega \tilde{n}_{N_3}\|_{L_{t,x}^{(d+1)/2}}. \quad (2.5)$$

We apply Proposition 2.8, (2.4) and the Sobolev inequality, then we have

$$\|\omega_1^{-1} \tilde{f}_N\|_{L_{t,x}^{2(d+1)/(d-1)}} \lesssim \langle N \rangle^{1/2-1} \|f_N\|_{V_{K_{\pm}}^2} = \langle N \rangle^{-1/2} \|f_N\|_{V_{K_{\pm}}^2}, \quad (2.6)$$

$$\begin{aligned} \|\omega \tilde{n}_{N_3}\|_{L_{t,x}^{(d+1)/2}} &\lesssim \| |\nabla_x|^{d(d-5)/2(d-1)} \omega \tilde{n}_{N_3} \|_{L_t^{(d+1)/2} L_x^{2(d^2-1)/(d^2-9)}} \\ &\lesssim \| |\nabla_x|^{d/2-2} \omega \tilde{n}_{N_3} \|_{V_{W_{\pm c}}^2} \end{aligned} \quad (2.7)$$

$$\lesssim N_3^{s_c+1} \|n_{N_3}\|_{V_{W_{\pm c}}^2} \quad (2.8)$$

Collecting (2.5), (2.6), (2.8) and $N_3 \lesssim N_1 \sim N_2$, we obtain

$$|I_1| \lesssim N_3^{s_c} \|u_{N_1}\|_{V_{K_{\pm}}^2} \|v_{N_2}\|_{V_{K_{\pm}}^2} \|n_{N_3}\|_{V_{W_{\pm c}}^2}.$$

Next, we prove (ii). For $d \geq 5$, by the Hölder inequality to have

$$|I_2| \lesssim \|\tilde{n}\|_{L_{t,x}^{(d+1)/2}} \|\omega_1^{-1} \tilde{v}\|_{L_{t,x}^{2(d+1)/(d-1)}} \|P_{<1} \tilde{u}\|_{L_{t,x}^{2(d+1)/(d-1)}}. \quad (2.9)$$

From Proposition 2.8, (2.7), Remark 2.2 and Lemma 2.10, we obtain

$$\|\tilde{n}\|_{L_{t,x}^{(d+1)/2}} \lesssim \|n\|_{\dot{Y}_{W_{\pm c}}^{s_c}}, \quad (2.10)$$

$$\|\omega_1^{-1} \tilde{v}\|_{L_{t,x}^{2(d+1)/(d-1)}} \lesssim \|\langle \nabla_x \rangle^{-1/2} v\|_{V_{K_{\pm}}^2} \lesssim \|\langle \nabla_x \rangle^{s_c} v\|_{V_{K_{\pm}}^2} \lesssim \|v\|_{Y_{K_{\pm}}^{s_c}}, \quad (2.11)$$

$$\|P_{<1} \tilde{u}\|_{L_{t,x}^{2(d+1)/(d-1)}} \lesssim \|\langle \nabla_x \rangle^{1/2} P_{<1} u\|_{V_{K_{\pm}}^2} \lesssim \|P_{<1} u\|_{V_{K_{\pm}}^2}. \quad (2.12)$$

Collecting (2.9)–(2.12), we obtain

$$|I_2| \lesssim \|n\|_{\dot{Y}_{W_{\pm c}}^{s_c}} \|v\|_{Y_{K_{\pm}}^{s_c}} \|P_{<1} u\|_{V_{K_{\pm}}^2}.$$

We prove (iii) for $d \geq 5$. We apply the Hölder inequality to have

$$|I_3| \lesssim \left\| \sum_{N_3 \lesssim N_2} \tilde{n}_{N_3} \right\|_{L_{t,x}^{(d+1)/2}} \|\omega_1^{-1} \tilde{v}_{N_2}\|_{L_{t,x}^{2(d+1)/(d-1)}} \|\tilde{u}_{N_1}\|_{L_{t,x}^{2(d+1)/(d-1)}}. \quad (2.13)$$

Similar to (2.7), the Sobolev inequality and Proposition 2.8, we have

$$\left\| \sum_{N_3 \lesssim N_2} \tilde{n}_{N_3} \right\|_{L_{t,x}^{(d+1)/2}} \lesssim \left\| |\nabla_x|^{s_c} \sum_{N_3 \lesssim N_2} \tilde{n}_{N_3} \right\|_{V_{W_{\pm c}}^2}. \quad (2.14)$$

By the L_x^2 orthogonality, we obtain

$$\begin{aligned} \left\| |\nabla_x|^{s_c} \sum_{N_3 \lesssim N_2} \tilde{n}_{N_3} \right\|_{V_{W_{\pm c}}^2}^2 &\lesssim \sup_{\{t_i\}_{i=0}^I \in \mathcal{Z}} \sum_{i=1}^I \sum_N N^{2s_c} \left\| P_N \left\{ W_{\pm c}(-t_i) \left(\sum_{N_3 \lesssim N_2} \tilde{n}_{N_3}(t_i) \right) \right. \right. \\ &\quad \left. \left. - W_{\pm c}(-t_{i-1}) \left(\sum_{N_3 \lesssim N_2} \tilde{n}_{N_3}(t_{i-1}) \right) \right\} \right\|_{L_x^2}^2. \end{aligned} \quad (2.15)$$

Since $P_N \tilde{n}_{N_3} = 0$ if $N_3 > 2N$ or $N_3 < N/2$ and P_N is projection, the right-hand side is bounded by

$$\begin{aligned} &\sup_{\{t_i\}_{i=0}^I \in \mathcal{Z}} \sum_{i=1}^I \sum_N N^{2s_c} \|W_{\pm c}(-t_i) P_N \tilde{n}(t_i) - W_{\pm c}(-t_{i-1}) P_N \tilde{n}(t_{i-1})\|_{L_x^2}^2 \\ &\lesssim \sum_N N^{2s_c} \sup_{\{t_i\}_{i=0}^I \in \mathcal{Z}} \|W_{\pm c}(-t_i) P_N \tilde{n}(t_i) - W_{\pm c}(-t_{i-1}) P_N \tilde{n}(t_{i-1})\|_{L_x^2}^2 \\ &\lesssim \|n\|_{\dot{Y}_{W_{\pm c}}^{s_c}}^2. \end{aligned} \quad (2.16)$$

Hence, from (2.13)–(2.16), (2.6) and $N_1 \sim N_2$, we have

$$\begin{aligned} |I_3| &\lesssim \|n\|_{\dot{Y}_{W_{\pm c}}^{s_c}} \langle N_2 \rangle^{-1/2} \|v_{N_2}\|_{V_{K_{\pm}}^2} \langle N_1 \rangle^{1/2} \|u_{N_1}\|_{V_{K_{\pm}}^2} \\ &\lesssim \|n\|_{\dot{Y}_{W_{\pm c}}^{s_c}} \|v_{N_2}\|_{V_{K_{\pm}}^2} \|u_{N_1}\|_{V_{K_{\pm}}^2}. \end{aligned}$$

We prove (iv). The estimate for I_5 is obtained by the same manner as the estimate for I_4 , so we only estimate I_4 . We apply the Hölder inequality to have

$$|I_4| \lesssim \|Q_{\geq M}^{W_{\pm c}} \tilde{n}_{N_3}\|_{L_{t,x}^2} \left\| \sum_{N_2 \ll N_1} Q_2 \omega_1^{-1} \tilde{v}_{N_2} \right\|_{L_{t,x}^{d+1}} \|Q_1 \tilde{u}_{N_1}\|_{L_{t,x}^{2(d+1)/(d-1)}}. \quad (2.17)$$

By Proposition 2.12, (2.6) and (2.4), we have

$$\|Q_{\geq M}^{W_{\pm c}} \tilde{n}_{N_3}\|_{L_{t,x}^2} \lesssim N_1^{-1/2} \|n_{N_3}\|_{V_{W_{\pm c}}^2}, \quad (2.18)$$

$$\|Q_1 \tilde{u}_{N_1}\|_{L_{t,x}^{2(d+1)/(d-1)}} \lesssim \langle N_1 \rangle^{1/2} \|u_{N_1}\|_{V_{K_{\pm}}^2}. \quad (2.19)$$

We apply the Sobolev inequality, Proposition 2.8, Proposition 2.12 and (2.4), we have

$$\begin{aligned} \left\| \sum_{N_2 \ll N_1} Q_2 \omega_1^{-1} \tilde{v}_{N_2} \right\|_{L_{t,x}^{d+1}} &\lesssim \left\| \langle \nabla_x \rangle^{d(d-3)/2(d-1)} \sum_{N_2 \ll N_1} Q_2 \omega_1^{-1} \tilde{v}_{N_2} \right\|_{L_t^{d+1} L_x^{2(d^2-1)/(d^2-5)}} \\ &\lesssim \left\| \langle \nabla_x \rangle^{d(d-3)/2(d-1)+1/(d-1)-1} \sum_{N_2 \ll N_1} \tilde{v}_{N_2} \right\|_{V_{K_{\pm}}^2}. \end{aligned} \quad (2.20)$$

Similar to (2.15) and (2.16), we have

$$\left\| \langle \nabla_x \rangle^{d(d-3)/2(d-1)+1/(d-1)-1} \sum_{N_2 \ll N_1} \tilde{v}_{N_2} \right\|_{V_{K\pm}^2} \lesssim \|v\|_{Y_{K\pm}^{sc}}. \quad (2.21)$$

Collecting (2.17)–(2.21) and $N_1 \gg 1$, we obtain

$$|I_4| \lesssim \|n_{N_3}\|_{V_{W\pm c}^2} \|v\|_{Y_{K\pm}^{sc}} \|u_{N_1}\|_{V_{K\pm}^2}.$$

□

The following proposition is in [27], Proposition 10.

Proposition 2.15. (*L^4 Strichartz estimate*) For all dyadic numbers $H \geq 1$ and N , it holds that

$$\|W_{\pm c}(t)P_N\phi\|_{L_{t,x}^4} \lesssim N^{(d-1)/4} \|P_N\phi\|_{L_x^2}, \quad \|K_{\pm}(t)P_H\varphi\|_{L_{t,x}^4} \lesssim H^{(d-1)/4} \|P_H\varphi\|_{L_x^2}.$$

From Proposition 2.5 and the above proposition, we obtain the following.

Proposition 2.16. For dyadic numbers $H \geq 1$ and N , it holds that

$$\|u_N\|_{L_{t,x}^4} \lesssim N^{(d-1)/4} \|u_N\|_{U_{W\pm c}^4}, \quad \|v_H\|_{L_{t,x}^4} \lesssim H^{(d-1)/4} \|v_H\|_{U_{K\pm}^4}.$$

Proposition 2.17. Let $u_M, v_N \in L^2(\mathbb{R}^{1+d})$ be such that

$$\text{supp } \mathcal{F}u_M \subset W_{L_1}^{\pm c} \cap (\mathbb{R} \times (C \cap P_M)), \quad \text{supp } \mathcal{F}v_N \subset KG_{L_2}^{\pm} \cap P_N$$

for dyadic numbers L_1, L_2, M, N and a cube $C \subset \mathbb{R}^d$ of side length e . If $L \ll M \sim N, c > 0$ and $c \neq 1$, it holds that

$$\|P_L(u_M v_N)\|_{L_{t,x}^2} \lesssim L^{(d-1)/2} (L_1 L_2)^{1/2} \|u_M\|_{L_{t,x}^2} \|v_N\|_{L_{t,x}^2}.$$

Proof. Let $f := \mathcal{F}u_M, g := \mathcal{F}v_N$. By the Cauchy-Schwarz inequality, we have

$$\left\| \int_{|\xi| \sim L} f(\tau_1, \xi_1) g(\tau - \tau_1, \xi - \xi_1) d\tau_1 d\xi_1 \right\|_{L_{\tau, \xi}^2} \lesssim \sup_{\tau, \xi} |E(\tau, \xi)|^{1/2} \|f\|_{L^2} \|g\|_{L^2}$$

where

$$E(\tau, \xi) = \{(\tau_1, \xi_1) \in \text{supp } f; (\tau - \tau_1, \xi - \xi_1) \in \text{supp } g, |\xi| \sim L\} \subset \mathbb{R}^{1+d}.$$

Put $\underline{L} := \min\{L_1, L_2\}, \bar{L} := \max\{L_1, L_2\}$. By the Fubini theorem,

$$|E(\tau, \xi)| \leq \underline{L} \left| \{\xi_1; |\tau \pm c|\xi_1| \pm |\xi - \xi_1| \lesssim \bar{L}, \xi_1 \in C, |\xi_1| \sim M, |\xi - \xi_1| \sim N, |\xi| \sim L\} \right|.$$

In the right-hand side of the above inequality, the subset of the ξ_1 is contained in a cube of side length m , where $m \sim \min\{e, N\} \sim L$. For some $i \in \{1, \dots, d\}$, we set $|(\xi - \xi_1)_i| \gtrsim N$, where $(\xi - \xi_1)_i$ denotes the i -th component of $\xi - \xi_1$. We compute

$$|\partial_{\xi_1, i}(\tau \pm c|\xi_1| \pm (1 + |\xi - \xi_1|))| = \left| \frac{(\xi - \xi_1)_i}{|\xi - \xi_1|} \pm c \frac{\xi_{1, i}}{|\xi_1|} \right|, \quad (2.22)$$

where $\xi_{1,i}$ be the i -th component of ξ_1 . Since $|(\xi - \xi_1)_i| \gtrsim N$ and $|\xi| \sim L$, it suffices to consider the case $|\xi_{0,i}| \ll |\xi_{1,i}|$, where $\xi_{0,i}$ be the i -th component of ξ . Firstly, we consider the case $0 < c \ll 1$. We have

$$r.h.s. \text{ of (2.22)} \geq \frac{|(\xi - \xi_1)_i|}{|\xi - \xi_1|} - c \frac{|\xi_{1,i}|}{|\xi_1|} \gtrsim 1 - c$$

from $|(\xi - \xi_1)_i| \gtrsim N \sim |\xi - \xi_1|$ and $|\xi_1| \geq |\xi_{1,i}|$. Secondly, we consider the case $c \sim 1, c \neq 1$. The assumption $L \ll N \sim M$ implies $(1 - \varepsilon)|\xi - \xi_1| \leq |\xi_1| \leq (1 + \varepsilon)|\xi - \xi_1|$ for sufficiently small $\varepsilon > 0$. From the above inequality and $|\xi_{0,i}| \ll |\xi_{1,i}|$, we obtain

$$r.h.s. \text{ of (2.22)} \gtrsim \left| c \frac{|\xi_{1,i}|}{|\xi_1|} - \frac{|(\xi - \xi_1)_i|}{|\xi - \xi_1|} \right| \gtrsim |c - 1|.$$

Finally, we consider the case $c \gg 1$. We have

$$r.h.s. \text{ of (2.22)} \gtrsim c \frac{|\xi_{1,i}|}{|\xi_1|} - \frac{|(\xi - \xi_1)_i|}{|\xi - \xi_1|} \gtrsim c - 1$$

since $|(\xi - \xi_1)_i| \gtrsim N$ and $|\xi_{0,i}| \ll |\xi_{1,i}|$. Therefore,

$$|\partial_{\xi_{1,i}}(\tau \pm c|\xi_1| \pm (1 + |\xi - \xi_1|))| \gtrsim |c - 1|. \quad (2.23)$$

Hence by (2.23) and the mean value theorem, we have

$$\begin{aligned} |\{\xi_1; |\tau \pm c|\xi_1| \pm |\xi - \xi_1| \} &\lesssim \bar{l}, \xi_1 \in C, |\xi_1| \sim M, |\xi - \xi_1| \sim N, |\xi| \sim L \} \\ &\lesssim |c - 1|^{-1} m^{d-1} \bar{l}. \end{aligned}$$

From $m \sim L$, we have

$$|E(\xi, \tau)|^{1/2} \lesssim (\bar{l} |c - 1|^{-1} m^{d-1} \bar{l})^{1/2} \sim |c - 1|^{-1/2} (L_1 L_2)^{1/2} L^{(d-1)/2}.$$

Thus, we obtain the result. \square

Proposition 2.17 implies the following.

Proposition 2.18. *Let $L \ll M \sim N, c > 0$ and $c \neq 1$. For $u_M = W_{\pm c}(t)P_M\phi, v_N = K_{\pm}(t)P_N\varphi$, it holds that*

$$\|P_L(u_M v_N)\|_{L_{t,x}^2} \lesssim L^{(d-1)/2} \|P_M\phi\|_{L_x^2} \|P_N\varphi\|_{L_x^2}.$$

From Proposition 2.5 and the above proposition, we have the following.

Proposition 2.19. *Let $L \ll M \sim N, c > 0$ and $c \neq 1$. It holds that*

$$\|P_L(u_M v_N)\|_{L_{t,x}^2} \lesssim L^{(d-1)/2} \|u_M\|_{U_{W_{\pm c}}^2} \|v_N\|_{U_{K_{\pm}}^2}.$$

The following proposition is in [8], Proposition 2.20.

Proposition 2.20. *Let $q > 1$, E be a Banach space, $A = K_{\pm}$ or $W_{\pm c}$ and $T : U_A^q \rightarrow E$ be a bounded, linear operator with $\|Tu\|_E \leq C_q \|u\|_{U_A^q}$ for all $u \in U_A^q$. In addition, assume that for some $1 \leq p < q$ there exists $C_p \in (0, C_q]$ such that the estimate $\|Tu\|_E \leq C_p \|u\|_{U_A^p}$ holds true for all $u \in U_A^p$. Then, T satisfies the estimate*

$$\|Tu\|_E \leq C_p (1 + \ln(C_q/C_p)) \|u\|_{V_A^p}, \quad u \in V_A^p.$$

Proposition 2.21. *Let $L \ll M \sim N$, $N \geq 1$, $c > 0$ and $c \neq 1$. For sufficiently small $\varepsilon > 0$, it holds that*

$$\|P_L(u_M v_N)\|_{L_{t,x}^2} \lesssim L^{(d-1)/2} (M/L)^{\varepsilon} \|u_M\|_{V_{W_{\pm c}}^2} \|v_N\|_{V_{K_{\pm}}^2}.$$

Proof. By the Hölder inequality, $M \sim N$, $N \geq 1$ and Proposition 2.16, we obtain

$$\|P_L(u_M v_N)\|_{L_{t,x}^2} \lesssim \|u_M\|_{L_{t,x}^4} \|v_N\|_{L_{t,x}^4} \lesssim M^{(d-1)/2} \|u_M\|_{U_{W_{\pm c}}^4} \|v_N\|_{U_{K_{\pm}}^4}. \quad (2.24)$$

Let $Sv := P_L(\tilde{P}_M u \tilde{P}_N v)$, where $\tilde{P}_M = P_{M/2} + P_M + P_{2M}$, such that $\tilde{P}_M P_M = P_M \cdot \tilde{P}_N$ is defined by the same manner as \tilde{P}_M . From (2.24) and $U_{W_{\pm c}}^2 \subset U_{W_{\pm c}}^4$, we have

$$\|S\|_{U_{K_{\pm}}^4 \rightarrow L^2} \lesssim M^{(d-1)/2} \|u\|_{U_{W_{\pm c}}^4} \lesssim M^{(d-1)/2} \|u\|_{U_{W_{\pm c}}^2}. \quad (2.25)$$

From Proposition 2.19, we have

$$\|S\|_{U_{K_{\pm}}^2 \rightarrow L^2} \lesssim L^{(d-1)/2} \|u\|_{U_{W_{\pm c}}^2}. \quad (2.26)$$

From (2.25), (2.26) and Proposition 2.20, for sufficiently small $\varepsilon' > 0$, we have

$$\|S\|_{V_{K_{\pm}}^2 \rightarrow L^2} \lesssim L^{(d-1)/2} (M/L)^{\varepsilon'} \|u\|_{U_{W_{\pm c}}^2}. \quad (2.27)$$

Let $Tu := P_L(\tilde{P}_M u \tilde{P}_N v)$. From Proposition 2.16, $M \sim N$ and $V_{K_{\pm}}^2 \subset U_{K_{\pm}}^4$, we have

$$\|T\|_{U_{W_{\pm c}}^4 \rightarrow L^2} \lesssim N^{(d-1)/2} \|v_N\|_{U_{K_{\pm}}^4} \lesssim N^{(d-1)/2} \|v_N\|_{V_{K_{\pm}}^2} \lesssim N^{(d-1)/2} \|v\|_{V_{K_{\pm}}^2}. \quad (2.28)$$

By (2.27), we have

$$\|T\|_{U_{W_{\pm c}}^2 \rightarrow L^2} \lesssim L^{(d-1)/2} (M/L)^{\varepsilon'} \|v\|_{V_{K_{\pm}}^2}. \quad (2.29)$$

Collecting (2.28), (2.29), $M \sim N$ and Proposition 2.20, we obtain

$$\|T\|_{V_{W_{\pm c}}^2 \rightarrow L^2} \lesssim L^{(d-1)/2} (M/L)^{2\varepsilon'} \|v\|_{V_{K_{\pm}}^2}.$$

Taking $\varepsilon = 2\varepsilon'$, the claim follows. \square

3. BILINEAR ESTIMATES

Proposition 3.1. *Let $d \geq 5$, $s = s_c = d/2 - 2$ and $c > 0, c \neq 1$. Then for all $0 < T < \infty$, it holds that*

$$\|I_{T,K_{\pm}}(n, v)\|_{Z_{K_{\pm}}^s} \lesssim \|n\|_{\dot{Y}_{W_{\pm c}}^{s_c}} \|v\|_{Y_{K_{\pm}}^s}, \quad (3.1)$$

$$\|I_{T,W_{\pm c}}(u, v)\|_{\dot{Z}_{W_{\pm c}}^s} \lesssim \|u\|_{Y_{K_{\pm}}^s} \|v\|_{Y_{K_{\pm}}^s}. \quad (3.2)$$

Remark 3.1. In (3.1) and (3.2), the implicit constant does not depend on T .

Proof. We denote $\tilde{u}_{N_1} := \mathbf{1}_{[0,T)} P_{N_1} u$, $\tilde{v}_{N_2} := \mathbf{1}_{[0,T)} P_{N_2} v$, $\tilde{n}_{N_3} := \mathbf{1}_{[0,T)} P_{N_3} n$. To prove (3.1), we need to estimate the following.

$$\|I_{T,K_{\pm}}(n, v)\|_{Z_{K_{\pm}}^{s_c}}^2 \lesssim \sum_{i=0}^3 J_i$$

where

$$\begin{aligned} J_0 &:= \left\| \int_0^t \mathbf{1}_{[0,T)}(t') K_{\pm}(t-t') P_{<1}(\tilde{n}(\omega_1^{-1} \tilde{v}))(t') dt' \right\|_{U_{K_{\pm}}^2}^2, \\ J_1 &:= \sum_{N_1 \geq 1} N_1^{2s_c} \left\| \int_0^t \mathbf{1}_{[0,T)}(t') K_{\pm}(t-t') \sum_{N_2 \sim N_1} \sum_{N_3 \lesssim N_2} P_{N_1}(\tilde{n}_{N_3}(\omega_1^{-1} \tilde{v}_{N_2}))(t') dt' \right\|_{U_{K_{\pm}}^2}^2, \\ J_2 &:= \sum_{N_1 \geq 1} N_1^{2s_c} \left\| \int_0^t \mathbf{1}_{[0,T)}(t') K_{\pm}(t-t') \sum_{N_2 \ll N_1} \sum_{N_3 \sim N_1} P_{N_1}(\tilde{n}_{N_3}(\omega_1^{-1} \tilde{v}_{N_2}))(t') dt' \right\|_{U_{K_{\pm}}^2}^2, \\ J_3 &:= \sum_{N_1 \geq 1} N_1^{2s_c} \left\| \int_0^t \mathbf{1}_{[0,T)}(t') K_{\pm}(t-t') \sum_{N_2 \gg N_1} \sum_{N_3 \sim N_2} P_{N_1}(\tilde{n}_{N_3}(\omega_1^{-1} \tilde{v}_{N_2}))(t') dt' \right\|_{U_{K_{\pm}}^2}^2. \end{aligned}$$

By Corollary 2.4 and Lemma 2.14 (ii), we have

$$\begin{aligned} J_0^{1/2} &\lesssim \sup_{\|u\|_{V_{K_{\pm}}^2}=1} \left| \int_{\mathbb{R}^{1+d}} \tilde{n}(\omega_1^{-1} \tilde{v})(\overline{P_{<1} \tilde{u}}) dx dt \right| \\ &\lesssim \|n\|_{\dot{Y}_{W_{\pm c}}^{s_c}} \|v\|_{Y_{K_{\pm}}^{s_c}}. \end{aligned} \quad (3.3)$$

We apply Corollary 2.4, $N_1 \sim N_2$, Lemma 2.14 (iii) and $\|\tilde{u}_{N_1}\|_{V_{K_{\pm}}^2} \lesssim \|u\|_{V_{K_{\pm}}^2}$, then

$$\begin{aligned} J_1 &\lesssim \sum_{N_1 \geq 1} N_1^{2s_c} \sup_{\|u\|_{V_{K_{\pm}}^2}=1} \left| \sum_{N_2 \sim N_1} \sum_{N_3 \lesssim N_2} \int_{\mathbb{R}^{1+d}} \tilde{n}_{N_3}(\omega_1^{-1} \tilde{v}_{N_2}) \overline{\tilde{u}_{N_1}} dx dt \right|^2 \\ &\lesssim \sum_{N_2 \gtrsim 1} N_2^{2s_c} \|n\|_{\dot{Y}_{W_{\pm c}}^{s_c}}^2 \|v_{N_2}\|_{V_{K_{\pm}}^2}^2 \\ &\lesssim \|n\|_{\dot{Y}_{W_{\pm c}}^{s_c}}^2 \|v\|_{Y_{K_{\pm}}^{s_c}}^2. \end{aligned} \quad (3.4)$$

For the estimate of J_2 , we take $M = \varepsilon N_1$ for sufficiently small $\varepsilon > 0$. Then, from Lemma 2.13, we have

$$\begin{aligned} & P_{N_1} Q_{<M}^{K_\pm} ((Q_{<M}^{W_{\pm c}} \tilde{n}_{N_3})(Q_{<M}^{K_\pm} \omega_1^{-1} \tilde{v}_{N_2})) \\ &= P_{N_1} Q_{<M}^{K_\pm} \left[\mathcal{F}^{-1} \left(\int_{\tau_1=\tau_2+\tau_3, \xi_1=\xi_2+\xi_3} (\widehat{Q_{<M}^{W_{\pm c}} \tilde{n}_{N_3}})(\tau_3, \xi_3) (\widehat{Q_{<M}^{K_\pm} \omega_1^{-1} \tilde{v}_{N_2}})(\tau_2, \xi_2) \right) \right] = 0 \end{aligned}$$

when $N_1 \gg \langle N_2 \rangle$. Therefore,

$$P_{N_1} (\tilde{n}_{N_3}(\omega_1^{-1} \tilde{v}_{N_2})) = \sum_{i=1}^3 P_{N_1} F_i,$$

where

$$\begin{aligned} F_1 &:= Q_1 ((Q_{\geq M}^{W_{\pm c}} \tilde{n}_{N_3})(Q_2 \omega_1^{-1} \tilde{v}_{N_2})), \quad F_2 := Q_1 ((Q_3 \tilde{n}_{N_3})(Q_{\geq M}^{K_\pm} \omega_1^{-1} \tilde{v}_{N_2})), \\ F_3 &:= Q_{\geq M}^{K_\pm} ((Q_3 \tilde{n}_{N_3})(Q_2 \omega_1^{-1} \tilde{v}_{N_2})). \end{aligned}$$

Here, $Q_1, Q_2 \in \{Q_{<M}^{K_\pm}, Q_{\geq M}^{K_\pm}\}$ and $Q_3 \in \{Q_{<M}^{W_{\pm c}}, Q_{\geq M}^{W_{\pm c}}\}$. For the estimate of F_1 , we apply Corollary 2.4, Lemma 2.11, Lemma 2.14 (iv), $N_3 \sim N_1 \geq 1$ and $\|\tilde{u}_{N_1}\|_{V_{K_\pm}^2} \lesssim \|u\|_{V_{K_\pm}^2}$, then we have

$$\begin{aligned} & \sum_{N_1 \geq 1} N_1^{2s_c} \left\| \int_0^t \mathbf{1}_{[0,T)}(t') K_\pm(t-t') \sum_{N_2 \ll N_1} \sum_{N_3 \sim N_1} P_{N_1} F_1(t') dt' \right\|_{U_{K_\pm}^2}^2 \\ & \lesssim \sum_{N_1 \geq 1} N_1^{2s_c} \sup_{\|u\|_{V_{K_\pm}^2}=1} \left| \sum_{N_2 \ll N_1} \sum_{N_3 \sim N_1} \int_{\mathbb{R}^{1+d}} (Q_{\geq M}^{W_{\pm c}} \tilde{n}_{N_3})(Q_2 \omega_1^{-1} \tilde{v}_{N_2})(\overline{Q_1 \tilde{u}_{N_1}}) dx dt \right|^2 \\ & \lesssim \sum_{N_3 \gtrsim 1} N_3^{2s_c} \|n_{N_3}\|_{V_{W_{\pm c}}^2}^2 \|v\|_{Y_{K_\pm}^{s_c}}^2 \\ & \lesssim \|n\|_{Y_{W_{\pm c}}^{s_c}}^2 \|v\|_{Y_{K_\pm}^{s_c}}^2. \end{aligned} \tag{3.5}$$

For the estimate of F_2 , we apply Corollary 2.4, Lemma 2.11 and the triangle inequality, we have

$$\begin{aligned} & \sum_{N_1 \geq 1} N_1^{2s_c} \left\| \int_0^t \mathbf{1}_{[0,T)}(t') K_\pm(t-t') \sum_{N_2 \ll N_1} \sum_{N_3 \sim N_1} P_{N_1} F_2(t') dt' \right\|_{U_{K_\pm}^2}^2 \\ & \lesssim \sum_{N_1 \geq 1} N_1^{2s_c} \sup_{\|u\|_{V_{K_\pm}^2}=1} \left| \sum_{N_2 \ll N_1} \sum_{N_3 \sim N_1} \int_{\mathbb{R}^{1+d}} (Q_3 \tilde{n}_{N_3})(Q_{\geq M}^{K_\pm} \omega_1^{-1} \tilde{v}_{N_2})(\overline{Q_1 \tilde{u}_{N_1}}) dx dt \right|^2 \\ & \lesssim \sum_{N_1 \geq 1} N_1^{2s_c} \sup_{\|u\|_{V_{K_\pm}^2}=1} \sum_{N_2 \ll N_1} \sum_{N_3 \sim N_1} \left| \int_{\mathbb{R}^{1+d}} (Q_3 \tilde{n}_{N_3})(Q_{\geq M}^{K_\pm} \omega_1^{-1} \tilde{v}_{N_2})(\overline{Q_1 \tilde{u}_{N_1}}) dx dt \right|^2. \end{aligned} \tag{3.6}$$

By Proposition 2.21, $N_2 \ll N_1 \sim N_3$, $N_1 \geq 1$ and Proposition 2.12, we have

$$\begin{aligned}
& \left| \int_{\mathbb{R}^{1+d}} (Q_3 \tilde{n}_{N_3}) (Q_{\geq M}^{K_{\pm}} \omega_1^{-1} \tilde{v}_{N_2}) (\overline{Q_1 \tilde{u}_{N_1}}) dx dt \right| \\
& \lesssim \|Q_{\geq M}^{K_{\pm}} \omega_1^{-1} \tilde{v}_{N_2}\|_{L_{t,x}^2} \|P_{N_2}((Q_3 \tilde{n}_{N_3})(\overline{Q_1 \tilde{u}_{N_1}}))\|_{L_{t,x}^2} \\
& \lesssim N_3^{-1/2} \langle N_2 \rangle^{-1} \|v_{N_2}\|_{V_{K_{\pm}}^2} N_2^{(d-1)/2} (N_3/N_2)^{\varepsilon} \|n_{N_3}\|_{V_{\pm c}^2} \|u_{N_1}\|_{V_{K_{\pm}}^2} \\
& \lesssim N_2^{s_c} (N_2/N_3)^{1/2-\varepsilon} \|v_{N_2}\|_{V_{K_{\pm}}^2} \|n_{N_3}\|_{V_{\pm c}^2} \|u_{N_1}\|_{V_{K_{\pm}}^2}. \tag{3.7}
\end{aligned}$$

By (3.7) and the Cauchy-Schwarz inequality, the right-hand side of (3.6) is bounded by

$$\begin{aligned}
& \sum_{N_3 \gtrsim 1} N_3^{2s_c} \|n_{N_3}\|_{V_{W_{\pm c}}^2}^2 \left(\sum_{N_2 \ll N_3} (N_2/N_3)^{1/2-\varepsilon} N_2^{s_c} \|v_{N_2}\|_{V_{K_{\pm}}^2} \right)^2 \\
& \lesssim \|n\|_{Y_{W_{\pm c}}^{s_c}}^2 \|v\|_{Y_{K_{\pm}}^{s_c}}^2. \tag{3.8}
\end{aligned}$$

For the estimate for F_3 , we apply Corollary 2.4, Lemma 2.11, Lemma 2.14 (iv), $N_3 \sim N_1 \geq 1$ and $\|\tilde{u}_{N_1}\|_{V_{K_{\pm}}^2} \lesssim \|u\|_{V_{K_{\pm}}^2}$, then we obtain

$$\begin{aligned}
& \sum_{N_1 \geq 1} N_1^{2s_c} \left\| \int_0^t \mathbf{1}_{[0,T)}(t') K_{\pm}(t-t') \sum_{N_2 \ll N_1} \sum_{N_3 \sim N_1} P_{N_1} F_3(t') dt' \right\|_{U_{K_{\pm}}^2}^2 \\
& \lesssim \sum_{N_1 \geq 1} N_1^{2s_c} \sup_{\|u\|_{V_{K_{\pm}}^2}=1} \left| \sum_{N_2 \ll N_1} \sum_{N_3 \sim N_1} \int_{\mathbb{R}^{1+d}} (Q_3 \tilde{n}_{N_3}) (Q_2 \omega_1^{-1} \tilde{v}_{N_2}) (\overline{Q_{\geq M}^{K_{\pm}} \tilde{u}_{N_1}}) dx dt \right|^2 \\
& \lesssim \sum_{N_3 \gtrsim 1} N_3^{2s_c} \|n_{N_3}\|_{V_{W_{\pm c}}^2}^2 \|v\|_{Y_{K_{\pm}}^{s_c}}^2 \\
& \lesssim \|n\|_{Y_{W_{\pm c}}^{s_c}}^2 \|v\|_{Y_{K_{\pm}}^{s_c}}^2. \tag{3.9}
\end{aligned}$$

Collecting (3.5), (3.8) and (3.9), we have

$$J_2 \lesssim \|n\|_{Y_{W_{\pm c}}^{s_c}}^2 \|v\|_{Y_{K_{\pm}}^{s_c}}^2. \tag{3.10}$$

By Corollary 2.4 and the triangle inequality to have

$$\begin{aligned}
J_3 & \lesssim \sum_{N_1 \geq 1} N_1^{2s_c} \sup_{\|u\|_{V_{K_{\pm}}^2}=1} \left| \sum_{N_2 \gg N_1} \sum_{N_3 \sim N_2} \int_{\mathbb{R}^{1+d}} \tilde{n}_{N_3} (\omega_1^{-1} \tilde{v}_{N_2}) \overline{\tilde{u}_{N_1}} dx dt \right|^2 \\
& \lesssim \sum_{N_1 \geq 1} N_1^{2s_c} \left(\sum_{N_2 \gg N_1} \sum_{N_3 \sim N_2} \sup_{\|u\|_{V_{K_{\pm}}^2}=1} \left| \int_{\mathbb{R}^{1+d}} \tilde{n}_{N_3} (\omega_1^{-1} \tilde{v}_{N_2}) \overline{\tilde{u}_{N_1}} dx dt \right| \right)^2. \tag{3.11}
\end{aligned}$$

By the same manner as the estimate for Lemma 2.14 (iii), we obtain

$$\left| \int_{\mathbb{R}^{1+d}} \tilde{n}_{N_3} (\omega_1^{-1} \tilde{v}_{N_2}) \overline{\tilde{u}_{N_1}} dx dt \right| \lesssim N_3^{s_c} \|n_{N_3}\|_{V_{W_{\pm c}}^2} \|v_{N_2}\|_{V_{K_{\pm}}^2} \|u_{N_1}\|_{V_{K_{\pm}}^2}. \tag{3.12}$$

From (3.12), the right-hand side of (3.11) is bounded by

$$\sum_{N_1 \geq 1} \left(\sum_{N_2 \gg N_1} \sum_{N_3 \sim N_2} N_1^{s_c} N_3^{s_c} \|n_{N_3}\|_{V_{W \pm c}^2} \|v_{N_2}\|_{V_{K \pm}^2} \right)^2.$$

From $s_c > 0$, $\|\cdot\|_{l^2 l^1} \lesssim \|\cdot\|_{l^1 l^2}$ and the Cauchy-Schwarz inequality, we have

$$\begin{aligned} J_3^{1/2} &\lesssim \sum_{N_2 \gtrsim 1} \sum_{N_3 \sim N_2} \left(\sum_{N_1 \ll N_2} N_1^{2s_c} N_3^{2s_c} \|n_{N_3}\|_{V_{W \pm c}^2}^2 \|v_{N_2}\|_{V_{K \pm}^2}^2 \right)^{1/2} \\ &\lesssim \sum_{N_2 \gtrsim 1} \sum_{N_3 \sim N_2} N_2^{s_c} N_3^{s_c} \|n_{N_3}\|_{V_{W \pm c}^2} \|v_{N_2}\|_{V_{K \pm}^2} \\ &\lesssim \|n\|_{\dot{Y}_{W \pm c}^{s_c}} \|v\|_{Y_{K \pm}^{s_c}}. \end{aligned} \quad (3.13)$$

Collecting (3.3), (3.4), (3.10) and (3.13), we obtain (3.1). We prove (3.2) below. By Corollary 2.4, we only need to estimate K_i ($i = 1, 2, 3$):

$$\begin{aligned} K_1 &:= \sum_{N_3} N_3^{2s_c} \sup_{\|n\|_{V_{W \pm c}^2} = 1} \left| \sum_{N_2 \sim N_3} \sum_{N_1 \ll N_3} \int_{\mathbb{R}^{1+d}} (\omega_1^{-1} \tilde{u}_{N_1}) (\overline{\omega_1^{-1} \tilde{v}_{N_2}}) (\overline{\omega \tilde{n}_{N_3}}) dx dt \right|^2, \\ K_2 &:= \sum_{N_3} N_3^{2s_c} \sup_{\|n\|_{V_{W \pm c}^2} = 1} \left| \sum_{N_2 \ll N_3} \sum_{N_1 \sim N_3} \int_{\mathbb{R}^{1+d}} (\omega_1^{-1} \tilde{u}_{N_1}) (\overline{\omega_1^{-1} \tilde{v}_{N_2}}) (\overline{\omega \tilde{n}_{N_3}}) dx dt \right|^2, \\ K_3 &:= \sum_{N_3} N_3^{2s_c} \sup_{\|n\|_{V_{W \pm c}^2} = 1} \left| \sum_{N_2 \gtrsim N_3} \sum_{N_1 \sim N_2} \int_{\mathbb{R}^{1+d}} (\omega_1^{-1} \tilde{u}_{N_1}) (\overline{\omega_1^{-1} \tilde{v}_{N_2}}) (\overline{\omega \tilde{n}_{N_3}}) dx dt \right|^2. \end{aligned}$$

First, we estimate K_1 . Put $K_1 = K_{1,1} + K_{1,2}$ where

$$\begin{aligned} K_{1,1} &:= \sum_{N_3 \lesssim 1} N_3^{2s_c} \sup_{\|n\|_{V_{W \pm c}^2} = 1} \left| \sum_{N_2 \sim N_3} \sum_{N_1 \ll N_3} \int_{\mathbb{R}^{1+d}} (\omega_1^{-1} \tilde{u}_{N_1}) (\overline{\omega_1^{-1} \tilde{v}_{N_2}}) \right. \\ &\quad \left. \times (\overline{\omega \tilde{n}_{N_3}}) dx dt \right|^2, \\ K_{1,2} &:= \sum_{N_3 \gg 1} N_3^{2s_c} \sup_{\|n\|_{V_{W \pm c}^2} = 1} \left| \sum_{N_2 \sim N_3} \sum_{N_1 \ll N_3} \int_{\mathbb{R}^{1+d}} (\omega_1^{-1} \tilde{u}_{N_1}) (\overline{\omega_1^{-1} \tilde{v}_{N_2}}) (\overline{\omega \tilde{n}_{N_3}}) dx dt \right|^2. \end{aligned} \quad (3.14)$$

By the same manner as the proof for Lemma (2.14) (i), we see

$$\begin{aligned} &\left| \int_{\mathbb{R}^{1+d}} \left(\sum_{N_1 \ll N_3} \omega_1^{-1} \tilde{u}_{N_1} \right) (\overline{\omega_1^{-1} \tilde{v}_{N_2}}) (\overline{\omega \tilde{n}_{N_3}}) dx dt \right| \\ &\lesssim \langle N_2 \rangle^{-1/2} \langle N_3 \rangle^{3/2} \|u\|_{Y_{K \pm}^{s_c}} \|v_{N_2}\|_{V_{K \pm}^2} \|n_{N_3}\|_{V_{W \pm c}^2}. \end{aligned} \quad (3.15)$$

Collecting (3.14), (3.15) and $N_2 \sim N_3 \lesssim 1$, we obtain

$$\begin{aligned}
K_{1,1} &\lesssim \sum_{N_2 \lesssim 1} N_2^{2s_c} (\|u\|_{Y_{K^\pm}^{s_c}} \langle N_2 \rangle^{-1/2+3/2} \|v_{N_2}\|_{V_{K^\pm}^2})^2 \\
&\lesssim \|u\|_{Y_{K^\pm}^{s_c}}^2 \sum_{N_2 \lesssim 1} N_2^{2s_c} \|v_{N_2}\|_{V_{K^\pm}^2}^2 \\
&\lesssim \|u\|_{Y_{K^\pm}^{s_c}}^2 \|v\|_{Y_{K^\pm}^{s_c}}^2.
\end{aligned}$$

For the estimate for $K_{1,2}$, we take $M = \varepsilon N_2$ for sufficiently small $\varepsilon > 0$. Then, from Lemma 2.13, we have

$$\begin{aligned}
&P_{N_1} Q_{<M}^{K^\pm} \omega_1^{-1} ((Q_{<M}^{K^\pm} \omega_1^{-1} \tilde{v}_{N_2})(Q_{<M}^{W^{\pm c}} \omega \tilde{n}_{N_3})) \\
&= P_{N_1} Q_{<M}^{K^\pm} \omega_1^{-1} \left[\mathcal{F}^{-1} \left(\int_{\tau_1=\tau_2+\tau_3, \xi_1=\xi_2+\xi_3} (\widehat{Q_{<M}^{K^\pm} \omega_1^{-1} \tilde{v}_{N_2}})(\tau_2, \xi_2) (\widehat{Q_{<M}^{W^{\pm c}} \omega \tilde{n}_{N_3}})(\tau_3, \xi_3) \right) \right] \\
&= 0
\end{aligned}$$

when $N_2 \gg \langle N_1 \rangle$. Therefore,

$$P_{N_1} ((\omega_1^{-1} \tilde{v}_{N_2})(\omega \tilde{n}_{N_3})) = \sum_{i=1}^3 P_{N_1} G_i,$$

where

$$\begin{aligned}
G_1 &:= Q_{\geq M}^{K^\pm} ((Q_2 \omega_1^{-1} \tilde{v}_{N_2})(Q_3 \omega \tilde{n}_{N_3})), & G_2 &:= Q_1 ((Q_{\geq M}^{K^\pm} \omega_1^{-1} \tilde{v}_{N_2})(Q_3 \omega \tilde{n}_{N_3})), \\
G_3 &:= Q_1 ((Q_2 \omega_1^{-1} \tilde{v}_{N_2})(Q_{\geq M}^{W^{\pm c}} \omega \tilde{n}_{N_3})).
\end{aligned}$$

Here, $Q_1, Q_2 \in \{Q_{<M}^{K^\pm}, Q_{\geq M}^{K^\pm}\}$ and $Q_3 \in \{Q_{<M}^{W^{\pm c}}, Q_{\geq M}^{W^{\pm c}}\}$. Hence, it follows that

$$K_{1,2} \leq \sum_{i=1}^3 K_{1,2,i}$$

where

$$K_{1,2,i} := \sum_{N_3 \gg 1} N_3^{2s_c} \sup_{\|n\|_{V_{W^{\pm c}}}^2 = 1} \left| \sum_{N_2 \sim N_3} \sum_{N_1 \ll N_3} \int_{\mathbb{R}^{1+d}} (\omega_1^{-1} \tilde{u}_{N_1}) \overline{G_i} dx dt \right|^2, \quad i = 1, 2, 3.$$

By Lemma 2.11, we have

$$K_{1,2,1} \lesssim \sum_{N_3 \gg 1} N_3^{2s_c} \sup_{\|n\|_{V_{W\pm c}^2}=1} \left| \sum_{N_2 \sim N_3} \sum_{N_1 \ll N_3} \int_{\mathbb{R}^{1+d}} (Q_{\geq M}^{K\pm} \omega_1^{-1} \tilde{u}_{N_1}) (\overline{Q_2 \omega_1^{-1} \tilde{v}_{N_2}}) \right. \\ \left. \times (\overline{Q_3 \omega \tilde{n}_{N_3}}) dx dt \right|^2, \quad (3.16)$$

$$K_{1,2,2} \lesssim \sum_{N_3 \gg 1} N_3^{2s_c} \sup_{\|n\|_{V_{W\pm c}^2}=1} \left| \sum_{N_2 \sim N_3} \sum_{N_1 \ll N_3} \int_{\mathbb{R}^{1+d}} (Q_1 \omega_1^{-1} \tilde{u}_{N_1}) (\overline{Q_{\geq M}^{K\pm} \omega_1^{-1} \tilde{v}_{N_2}}) \right. \\ \left. \times (\overline{Q_3 \omega \tilde{n}_{N_3}}) dx dt \right|^2, \quad (3.17)$$

$$K_{1,2,3} \lesssim \sum_{N_3 \gg 1} N_3^{2s_c} \sup_{\|n\|_{V_{W\pm c}^2}=1} \left| \sum_{N_2 \sim N_3} \sum_{N_1 \ll N_3} \int_{\mathbb{R}^{1+d}} (Q_1 \omega_1^{-1} \tilde{u}_{N_1}) (\overline{Q_2 \omega_1^{-1} \tilde{v}_{N_2}}) \right. \\ \left. \times (\overline{Q_{\geq M}^{W\pm c} \omega \tilde{n}_{N_3}}) dx dt \right|^2. \quad (3.18)$$

By the same manner as the estimate for F_2 , we apply Proposition 2.21, $N_1 \ll N_2 \sim N_3$, $N_3 \gg 1$ and Proposition 2.12, then we obtain

$$\left| \int_{\mathbb{R}^{1+d}} (Q_{\geq M}^{K\pm} \omega_1^{-1} \tilde{u}_{N_1}) (\overline{Q_2 \omega_1^{-1} \tilde{v}_{N_2}}) (\overline{Q_3 \omega \tilde{n}_{N_3}}) dx dt \right| \\ \lesssim \|Q_{\geq M}^{K\pm} \omega_1^{-1} \tilde{u}_{N_1}\|_{L_{t,x}^2} \|P_{N_1}((\overline{Q_2 \omega_1^{-1} \tilde{v}_{N_2}})(\overline{Q_3 \omega \tilde{n}_{N_3}}))\|_{L_{t,x}^2} \\ \lesssim N_3^{-1/2} \langle N_1 \rangle^{-1} \|u_{N_1}\|_{V_{K\pm}^2} N_1^{(d-1)/2} (N_3/N_1)^\varepsilon \langle N_2 \rangle^{-1} \|v_{N_2}\|_{V_{K\pm}^2} N_3 \|n_{N_3}\|_{V_{W\pm c}^2} \\ \lesssim N_1^{s_c} (N_1/N_3)^{1/2-\varepsilon} \langle N_2 \rangle^{-1} N_3 \|u_{N_1}\|_{V_{K\pm}^2} \|v_{N_2}\|_{V_{K\pm}^2} \|n_{N_3}\|_{V_{W\pm c}^2}. \quad (3.19)$$

From (3.16), (3.19), $N_3 \gg 1$, $N_2 \sim N_3$ and the Cauchy-Schwarz inequality, we have

$$K_{1,2,1} \lesssim \sum_{N_2 \gg 1} N_2^{2s_c} \left(\sum_{N_1 \ll N_2} N_1^{s_c} \|u_{N_1}\|_{V_{K\pm}^2} (N_1/N_2)^{1/2-\varepsilon} \langle N_2 \rangle^{-1} N_2 \|v_{N_2}\|_{V_{K\pm}^2} \right)^2 \\ \lesssim \|u\|_{Y_{K\pm}^{s_c}}^2 \|v\|_{Y_{K\pm}^{s_c}}^2.$$

By Lemma 2.14 (iv), $i = 5$, we obtain

$$\left| \int_{\mathbb{R}^{1+d}} \left(\sum_{N_1 \ll N_3} Q_1 \omega_1^{-1} \tilde{u}_{N_1} \right) (\overline{Q_{\geq M}^{K\pm} \omega_1^{-1} \tilde{v}_{N_2}}) (\overline{Q_3 \omega \tilde{n}_{N_3}}) dx dt \right| \\ \lesssim \langle N_2 \rangle^{-1} N_3 \|u\|_{Y_{K\pm}^{s_c}} \|v_{N_2}\|_{V_{K\pm}^2} \|n_{N_3}\|_{V_{W\pm c}^2}. \quad (3.20)$$

From (3.17), (3.20), $N_3 \gg 1$ and $N_2 \sim N_3$, we have

$$K_{1,2,2} \lesssim \sum_{N_2 \gg 1} N_2^{2s_c} (\|u\|_{Y_{K\pm}^{s_c}} \|v_{N_2}\|_{V_{K\pm}^2})^2 \lesssim \|u\|_{Y_{K\pm}^{s_c}}^2 \|v\|_{Y_{K\pm}^{s_c}}^2.$$

By Lemma 2.14 (iv), $i = 4$, we obtain

$$\begin{aligned} & \left| \int_{\mathbb{R}^{1+d}} \left(\sum_{N_1 \ll N_3} Q_1 \omega_1^{-1} \tilde{u}_{N_1} \right) (\overline{Q_2 \omega_1^{-1} \tilde{v}_{N_2}}) (\overline{Q_{\geq M}^{W_{\pm c}} \omega \tilde{n}_{N_3}}) dx dt \right| \\ & \lesssim \langle N_2 \rangle^{-1} N_3 \|u\|_{Y_{K_{\pm}}^{sc}} \|v_{N_2}\|_{V_{K_{\pm}}^2} \|n_{N_3}\|_{V_{W_{\pm c}}^2}. \end{aligned} \quad (3.21)$$

From (3.18), (3.21), $N_3 \gg 1$ and $N_2 \sim N_3$, we have

$$K_{1,2,3} \lesssim \sum_{N_2 \gg 1} N_2^{2sc} (\|u\|_{Y_{K_{\pm}}^{sc}} \|v_{N_2}\|_{V_{K_{\pm}}^2})^2 \lesssim \|u\|_{Y_{K_{\pm}}^{sc}}^2 \|v\|_{Y_{K_{\pm}}^{sc}}^2.$$

By symmetry, the estimate for K_2 is obtained by the same manner as the estimate for K_1 . Hence, we omit the estimate for K_2 . By the triangle inequality, Lemma 2.14 (i) and the Cauchy-Schwarz inequality, we have

$$\begin{aligned} K_3^{1/2} & \lesssim \sum_{N_2} \sum_{N_1 \sim N_2} \left\{ \sum_{N_3 \lesssim N_2} N_3^{2sc} \sup_{\|n\|_{V_{W_{\pm c}}^2} = 1} \left| \int_{\mathbb{R}^{1+d}} (\omega_1^{-1} \tilde{u}_{N_1}) (\overline{\omega_1^{-1} \tilde{v}_{N_2}}) (\overline{\omega \tilde{n}_{N_3}}) dx dt \right|^2 \right\}^{1/2} \\ & \lesssim \sum_{N_2} \sum_{N_1 \sim N_2} \left\{ \sum_{N_3 \lesssim N_2} N_3^{2sc} (N_3^{sc} \|u_{N_1}\|_{V_{K_{\pm}}^2} \|v_{N_2}\|_{V_{K_{\pm}}^2})^2 \right\}^{1/2} \\ & \lesssim \sum_{N_2} \sum_{N_1 \sim N_2} N_1^{sc} N_2^{sc} \|u_{N_1}\|_{V_{K_{\pm}}^2} \|v_{N_2}\|_{V_{K_{\pm}}^2} \\ & \lesssim \|u\|_{Y_{K_{\pm}}^{sc}} \|v\|_{Y_{K_{\pm}}^{sc}}. \end{aligned}$$

Therefore, we obtain (3.2). \square

4. THE PROOF OF THE MAIN THEOREM

We define

$$u_{\pm} := \omega_1 u \pm i \partial_t u, \quad n_{\pm} := n \pm i(c\omega)^{-1} \partial_t n$$

where $\omega_1 := (1 - \Delta)^{1/2}$, $\omega := (-\Delta)^{1/2}$. Then the wave equation in (1.1) is rewritten into

$$\begin{cases} i \partial_t u_{\pm} \mp \omega_1 u_{\pm} = \pm(1/4)(n_+ + n_-)(\omega_1^{-1} u_+ + \omega_1^{-1} u_-), & (t, x) \in [-T, T] \times \mathbb{R}^d, \\ i \partial_t n_{\pm} \mp c \omega n_{\pm} = \pm(4c)^{-1} \omega |\omega_1^{-1} u_+ + \omega_1^{-1} u_-|^2, & (t, x) \in [-T, T] \times \mathbb{R}^d, \\ (u_{\pm}, n_{\pm})|_{t=0} = (u_{\pm 0}, n_{\pm 0}) \in H^s(\mathbb{R}^d) \times \dot{H}^s(\mathbb{R}^d). \end{cases} \quad (4.1)$$

Hence by the Duhamel principle, we consider the following integral equation corresponding to (4.1) on the time interval $[0, T]$ with $0 < T \leq \infty$:

$$u_{\pm} = \Phi_1(u_{\pm}, n_+, n_-), \quad n_{\pm} = \Phi_2(n_{\pm}, u_+, u_-), \quad (4.2)$$

where

$$\begin{aligned}\Phi_1(u_{\pm}, n_{\pm}, n_{\pm}) &:= K_{\pm}(t)u_{\pm 0} \pm (1/4)\{I_{T,K_{\pm}}(n_{+}, u_{+})(t) + I_{T,K_{\pm}}(n_{+}, u_{-})(t) \\ &\quad + I_{T,K_{\pm}}(n_{-}, u_{+})(t) + I_{T,K_{\pm}}(n_{-}, u_{-})(t)\}, \\ \Phi_2(n_{\pm}, u_{\pm}, u_{\pm}) &:= W_{\pm c}(t)n_{\pm 0} \pm (4c)^{-1}\{I_{T,W_{\pm c}}(u_{+}, u_{+})(t) + I_{T,W_{\pm c}}(u_{+}, u_{-})(t) \\ &\quad + I_{T,W_{\pm c}}(u_{-}, u_{+})(t) + I_{T,W_{\pm c}}(u_{-}, u_{-})(t)\}.\end{aligned}$$

Proposition 4.1. (i) Let $d \geq 5, s = s_c = d/2 - 2$ and $\delta > 0$ be sufficiently small. For all $(u_{\pm 0}, n_{\pm 0}) \in B_{\delta}(H^s(\mathbb{R}^d) \times \dot{H}^s(\mathbb{R}^d))$ and for all $0 < T < \infty$, there exists a unique solution of (4.2) on $[0, T]$ such that

$$(u_{\pm}, n_{\pm}) \in Y_{K_{\pm}}^s([0, T]) \times \dot{Y}_{W_{\pm c}}^s([0, T]) \subset C([0, T]; H^s(\mathbb{R}^d)) \times C([0, T]; \dot{H}^s(\mathbb{R}^d)).$$

(ii) The flow map obtained by (i):

$B_{\delta}(H^s(\mathbb{R}^d)) \times B_{\delta}(\dot{H}^s(\mathbb{R}^d)) \ni (u_{\pm 0}, n_{\pm 0}) \mapsto (u_{\pm}, n_{\pm}) \in Y_{K_{\pm}}^s([0, T]) \times \dot{Y}_{W_{\pm c}}^s([0, T])$ is Lipschitz continuous.

Remark 4.1. Due to the time reversibility of the Klein-Gordon-Zakharov equation, Proposition 4.1 also holds in corresponding time interval $[-T, 0]$

Remark 4.2. By (i) in Proposition 4.1 and Remark 4.1, for any $T > 0$, we have solutions to (4.2) $(u_{\pm}(t), n_{\pm}(t))$ on $[0, T]$ and $[-T, 0]$. If initial data $(u_{\pm 0}, n_{\pm 0}) \in B_{\delta}(H^s(\mathbb{R}^d) \times \dot{H}^s(\mathbb{R}^d))$, then we can take T arbitrary large and by uniqueness, $(u_{\pm}(t), n_{\pm}(t)) \in C((-\infty, \infty); H^s(\mathbb{R}^d)) \times C((-\infty, \infty); \dot{H}^s(\mathbb{R}^d))$ can be defined uniquely.

Proposition 4.2. Let the solution $(u_{\pm}(t), n_{\pm}(t))$ to (4.2) on $(-\infty, \infty)$ obtained by Proposition 4.1, Remark 4.1 and Remark 4.2 with initial data $(u_{\pm 0}, n_{\pm 0}) \in B_{\delta}(H^s(\mathbb{R}^d) \times \dot{H}^s(\mathbb{R}^d))$. Then, there exist $(u_{\pm, +\infty}, n_{\pm, +\infty})$ and $(u_{\pm, -\infty}, n_{\pm, -\infty})$ in $H^s(\mathbb{R}^d) \times \dot{H}^s(\mathbb{R}^d)$ such that

$$\begin{aligned}\lim_{t \rightarrow +\infty} (\|u_{\pm}(t) - K_{\pm}(t)u_{\pm, +\infty}\|_{H_x^s(\mathbb{R}^d)} + \|n_{\pm}(t) - W_{\pm c}(t)n_{\pm, +\infty}\|_{\dot{H}_x^s(\mathbb{R}^d)}) &= 0, \\ \lim_{t \rightarrow -\infty} (\|u_{\pm}(t) - K_{\pm}(t)u_{\pm, -\infty}\|_{H_x^s(\mathbb{R}^d)} + \|n_{\pm}(t) - W_{\pm c}(t)n_{\pm, -\infty}\|_{\dot{H}_x^s(\mathbb{R}^d)}) &= 0.\end{aligned}$$

proof of Proposition 4.1. First, we prove (i). By Proposition 2.8, there exists $C > 0$ such that

$$\|K_{\pm}(t)u_{\pm 0}\|_{Y_{K_{\pm}}^s} \leq C\|u_{\pm 0}\|_{H^s}, \quad \|W_{\pm c}(t)n_{\pm 0}\|_{\dot{Y}_{W_{\pm c}}^s} \leq C\|n_{\pm 0}\|_{\dot{H}^s}.$$

We denote time interval $I := [0, T]$. If $(u_{\pm 0}, n_{\pm 0}) \in B_{\delta}(H^s(\mathbb{R}^d) \times \dot{H}^s(\mathbb{R}^d))$ is small and $(u_{\pm}, n_{\pm}) \in B_r(Y_{K_{\pm}}^s(I) \times \dot{Y}_{W_{\pm c}}^s(I))$, $s = d/2 - 2$, then by Proposition 3.1 and

Remark 3.1, we have

$$\begin{aligned}
& \|\Phi_1(u_\pm, n_+, n_-)\|_{Y_{K_\pm}^s(I)} \\
& \leq C\delta + (C/4)(\|n_+\|_{\dot{Y}_{W_{+c}}^s(I)}\|u_+\|_{Y_{K_+}^s(I)} + \|n_+\|_{\dot{Y}_{W_{+c}}^s(I)}\|u_-\|_{Y_{K_-}^s(I)} \\
& \quad + \|n_-\|_{\dot{Y}_{W_{-c}}^s(I)}\|u_+\|_{Y_{K_+}^s(I)} + \|n_-\|_{\dot{Y}_{W_{-c}}^s(I)}\|u_-\|_{Y_{K_-}^s(I)}), \\
& \|\Phi_2(n_\pm, u_+, u_-)\|_{\dot{Y}_{W_{\pm c}}^s(I)} \\
& \leq C\delta + (C/4c)(\|u_+\|_{Y_{K_+}^s(I)}^2 + 2\|u_+\|_{Y_{K_+}^s(I)}\|u_-\|_{Y_{K_-}^s(I)} + \|u_-\|_{Y_{K_-}^s(I)}^2).
\end{aligned}$$

Taking $\delta = r^2$ and $r = \min\{1, c\}/(4C)$, then we have

$$\|\Phi_1(u_\pm, n_+, n_-)\|_{Y_{K_\pm}^s(I)} \leq r, \quad \|\Phi_2(n_\pm, u_+, u_-)\|_{\dot{Y}_{W_{\pm c}}^s(I)} \leq r.$$

Hence, (Φ_1, Φ_2) is a map from $B_r(Y_{K_\pm}^s([0, T]) \times \dot{Y}_{W_{\pm c}}^s([0, T]))$ into itself. If we also assume $(v_\pm, m_\pm) \in B_r(Y_{K_\pm}^s(I) \times \dot{Y}_{W_{\pm c}}^s(I))$, then we have

$$\begin{aligned}
& \|\Phi_1(u_\pm, n_+, n_-) - \Phi_1(v_\pm, m_+, m_-)\|_{Y_{K_\pm}^s(I)} \\
& \leq (1/8)(\|u_+ - v_+\|_{Y_{K_+}^s(I)} + \|u_- - v_-\|_{Y_{K_-}^s(I)} \\
& \quad + \|n_+ - m_+\|_{\dot{Y}_{W_{+c}}^s(I)} + \|n_- - m_-\|_{\dot{Y}_{W_{-c}}^s(I)}), \tag{4.3}
\end{aligned}$$

$$\begin{aligned}
& \|\Phi_2(n_\pm, u_+, u_-) - \Phi_2(m_\pm, v_+, v_-)\|_{\dot{Y}_{W_{\pm c}}^s(I)} \\
& \leq (1/4)(\|u_+ - v_+\|_{Y_{K_+}^s(I)} + \|u_- - v_-\|_{Y_{K_-}^s(I)}). \tag{4.4}
\end{aligned}$$

Thus, (Φ_1, Φ_2) is a contraction mapping on $B_r(Y_{K_\pm}^s([0, T]) \times \dot{Y}_{W_{\pm c}}^s([0, T]))$. Hence, by the Banach fixed point theorem, we have a solution to (4.2) in it. We assume that $(u_\pm(0), n_\pm(0)), (v_\pm(0), m_\pm(0))$ are both small and $s = d/2 - 2$ for $d \geq 5$. Let $(u_\pm, n_\pm), (v_\pm, m_\pm) \in Y_{K_\pm}^s([0, T]) \times \dot{Y}_{W_{\pm c}}^s([0, T])$ are two solutions satisfying $(u_\pm(0), n_\pm(0)) = (v_\pm(0), m_\pm(0))$. Moreover,

$$T' := \sup\{0 \leq t \leq T; u_\pm(t) = v_\pm(t), n_\pm(t) = m_\pm(t)\} < T.$$

By a translation in t , it suffices to consider $T' = 0$. Let $0 < \tau \leq T$ be fixed later. From (4.3)–(4.4) and Proposition 2.9, we obtain

$$\begin{aligned}
& \|u_\pm - v_\pm\|_{Y_{K_\pm}^s([0, \tau])} \\
& \leq (1/7)(\|n_+ - m_+\|_{\dot{Y}_{W_{+c}}^s([0, \tau])} + \|n_- - m_-\|_{\dot{Y}_{W_{-c}}^s([0, \tau])} + \|u_+ - v_+\|_{Y_{K_+}^s([0, \tau])}), \tag{4.5}
\end{aligned}$$

$$\|n_\pm - m_\pm\|_{\dot{Y}_{W_{\pm c}}^s([0, \tau])} \leq (1/4)(\|u_+ - v_+\|_{Y_{K_+}^s([0, \tau])} + \|u_- - v_-\|_{Y_{K_-}^s([0, \tau])}). \tag{4.6}$$

From (4.5) and (4.6), we obtain

$$u_\pm = v_\pm, \quad n_\pm = m_\pm$$

on $[0, \tau]$ if $0 < \tau \leq T$ be sufficiently small. This contradicts the definition of T' . Therefore, the uniqueness of the solution (u_{\pm}, n_{\pm}) is showed. (ii) follows from the standard argument, so we omit the proof. \square

Finally, we prove Proposition 4.2. The proof is the same manner as the proof for Proposition 4.2 in [14].

Proof. There exists $M > 0$ such that for all $0 < T < \infty$,

$$\begin{aligned} \|u_{\pm}\|_{Y_{K_{\pm}}^s([0,T])} + \|n_{\pm}\|_{\dot{Y}_{W_{\pm c}}^s([0,T])} &< M, \\ \|u_{\pm}\|_{Y_{K_{\pm}}^s([-T,0])} + \|n_{\pm}\|_{\dot{Y}_{W_{\pm c}}^s([-T,0])} &< M \end{aligned}$$

holds since r in the proof of Proposition 4.1 does not depend on T . Take $\{t_k\}_{k=0}^K \in \mathcal{Z}_0$ and $0 < T < \infty$ such that $-T < t_0, t_K < T$. By L_x^2 orthogonality,

$$\begin{aligned} &\left(\sum_{k=1}^K \|\langle \nabla_x \rangle^s (K_{\pm}(-t_k)u_{\pm}(t_k) - K_{\pm}(-t_{k-1})u_{\pm}(t_{k-1}))\|_{L_x^2}^2 \right)^{1/2} \\ &\lesssim \|\langle \nabla_x \rangle^s u_{\pm}\|_{V_{K_{\pm}}^2([0,T])} + \|\langle \nabla_x \rangle^s u_{\pm}\|_{V_{K_{\pm}}^2([-T,0])} \\ &\lesssim \|u_{\pm}\|_{Y_{K_{\pm}}^s([0,T])} + \|u_{\pm}\|_{Y_{K_{\pm}}^s([-T,0])} \\ &< 2M. \end{aligned}$$

Thus,

$$\sup_{\{t_k\}_{k=0}^K \in \mathcal{Z}_0} \left(\sum_{k=1}^K \|\langle \nabla_x \rangle^s K_{\pm}(-t_k)u_{\pm}(t_k) - \langle \nabla_x \rangle^s K_{\pm}(-t_{k-1})u_{\pm}(t_{k-1})\|_{L_x^2}^2 \right)^{1/2} \lesssim M.$$

Hence, there exists $f_{\pm} := \lim_{t \rightarrow \pm\infty} \langle \nabla_x \rangle^s K_{\pm}(-t)u_{\pm}(t)$ in $L_x^2(\mathbb{R}^d)$. Then put $u_{\pm\infty} := \langle \nabla_x \rangle^{-s} f_{\pm}$, we obtain

$$\|\langle \nabla_x \rangle^s K_{\pm}(-t)u_{\pm}(t) - f_{\pm}\|_{L_x^2} = \|u_{\pm}(t) - K_{\pm}(t)u_{\pm\infty}\|_{H_x^s} \rightarrow 0$$

as $t \rightarrow \pm\infty$. The scattering result for the wave equation is obtained similarly. \square

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